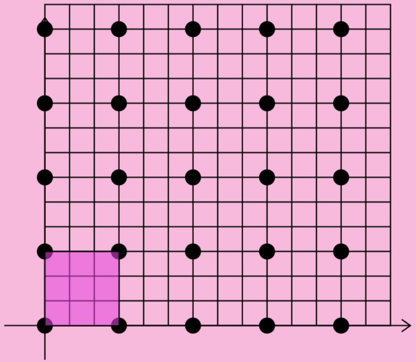
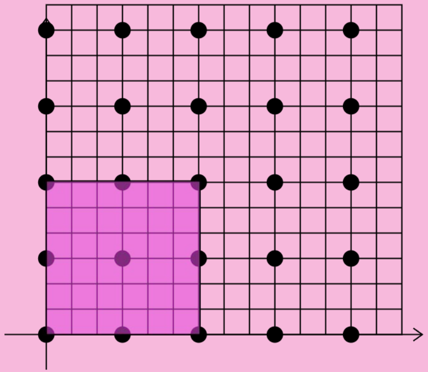
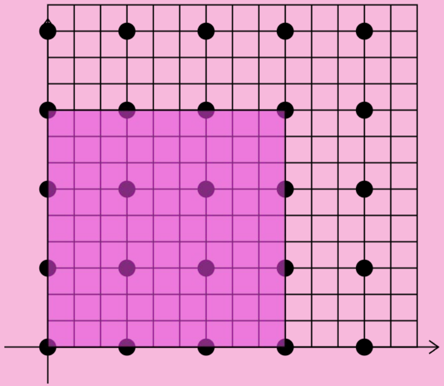




Invitation to Ehrhart Theory

Discrete volume and more!

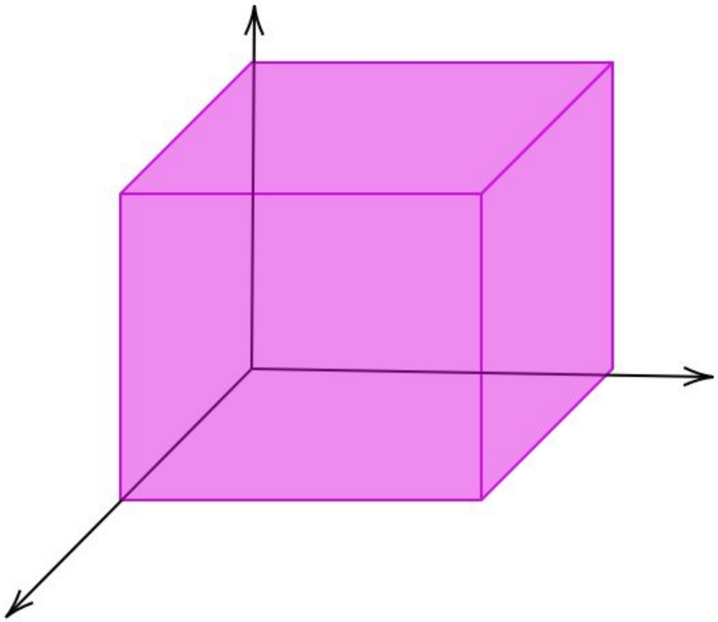
Plan for today!

- What is a polytope?
- Where do they show up?
- Volume computations: continuous and discrete
- Leading to: Ehrhart polynomial
- What kinds of questions do people ask in Ehrhart theory?

What is a polytope?

2 equivalent definitions:

- Convex hull of a finite set of points
- Intersection of a finite set of linear inequalities



What is a polytope?

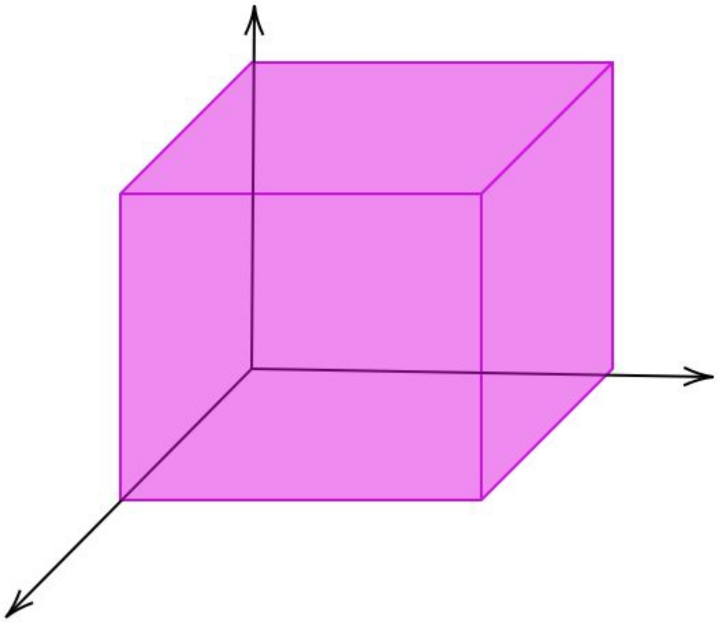
2 equivalent definitions:

- Convex hull of a finite set of points

  (0,0,0), (1,0,0), (0,1,0), (0,0,1)

  (1,1,0), (0,1,1), (1,0,1), (1,1,1)

- Intersection of a finite set of linear inequalities



What is a polytope?

2 equivalent definitions:

- Convex hull of a finite set of points

  (0,0,0), (1,0,0), (0,1,0), (0,0,1)

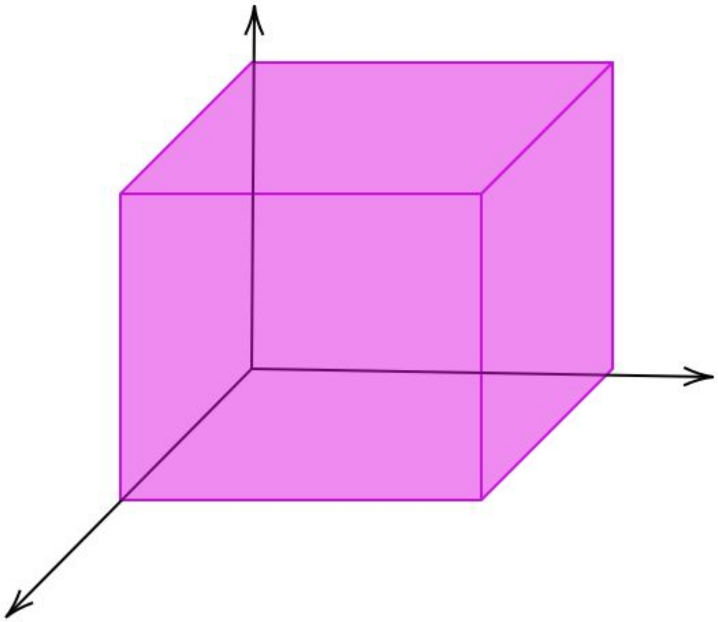
  (1,1,0), (0,1,1), (1,0,0), (1,1,1)

- Intersection of a finite set of linear inequalities

  $0 \leq x \leq 1$

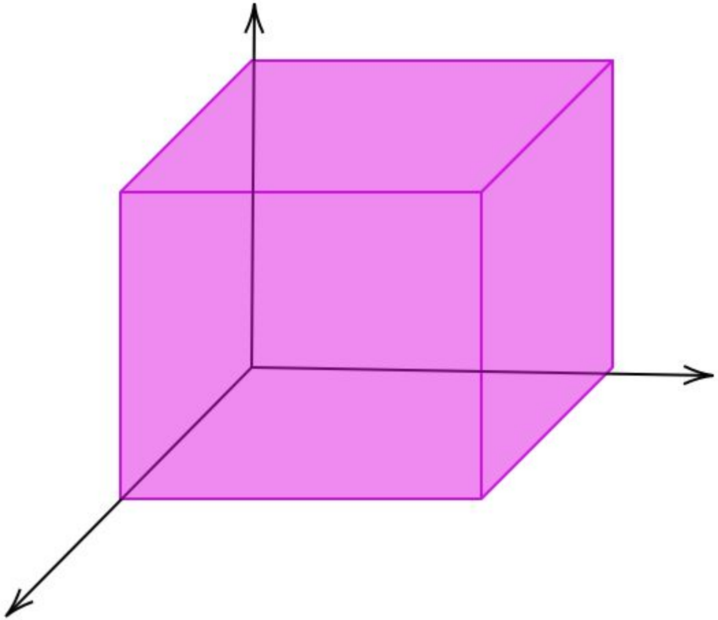
  $0 \leq y \leq 1$

  $0 \leq z \leq 1$

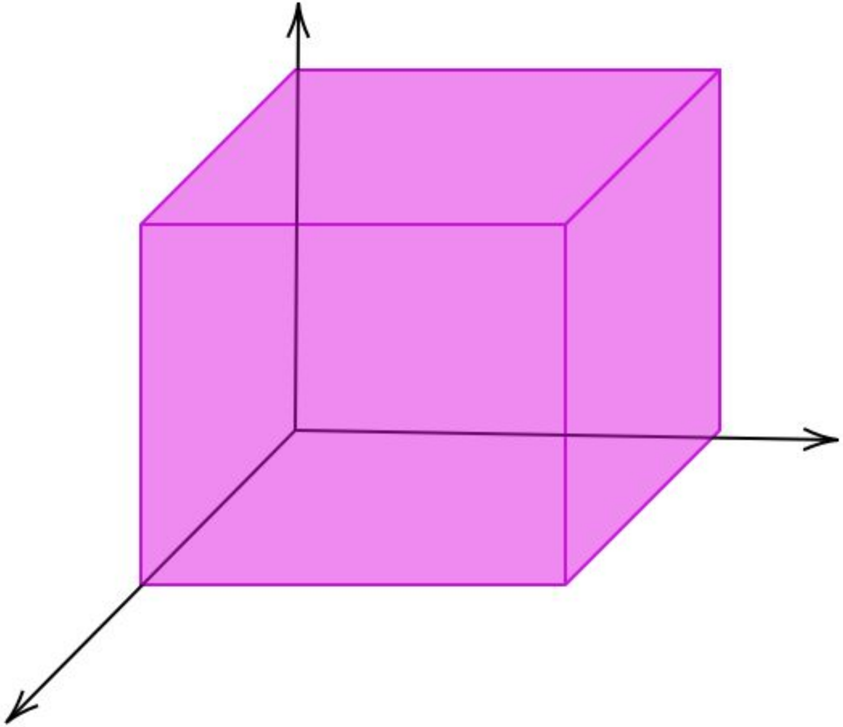


What is a polytope?

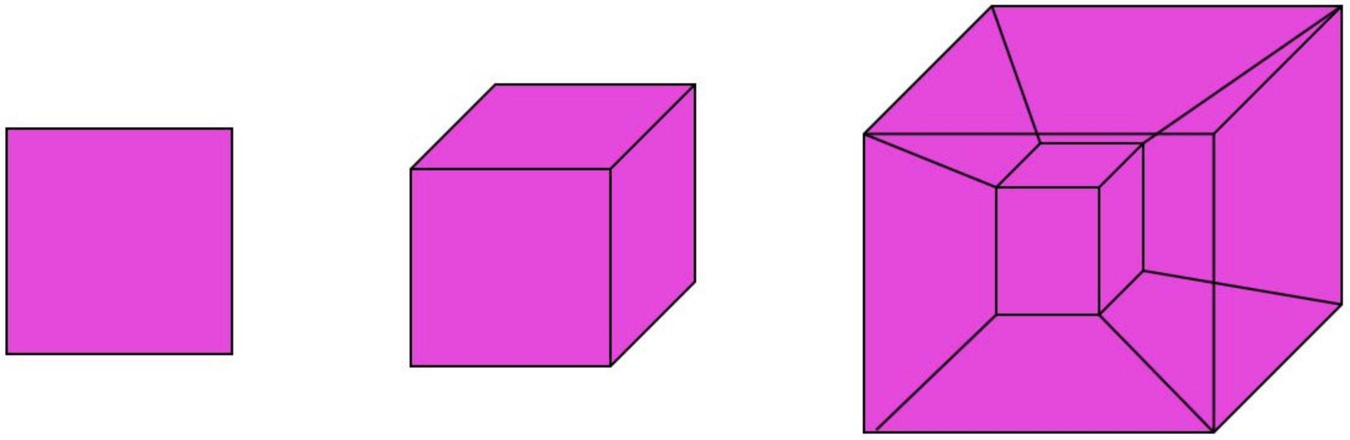
Alicia Boole Stott



Anatomy of a polytope



Cubes… in any dimension!



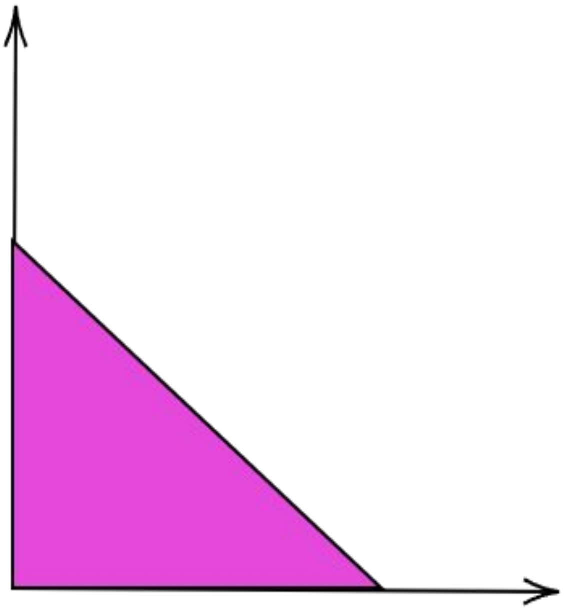
Where in the world can we find polytopes?

- In this room!
- Linear inequalities are everywhere… for example optimization!
- Hidden in other fields of mathematics!
- Bridge between __________ + _______________

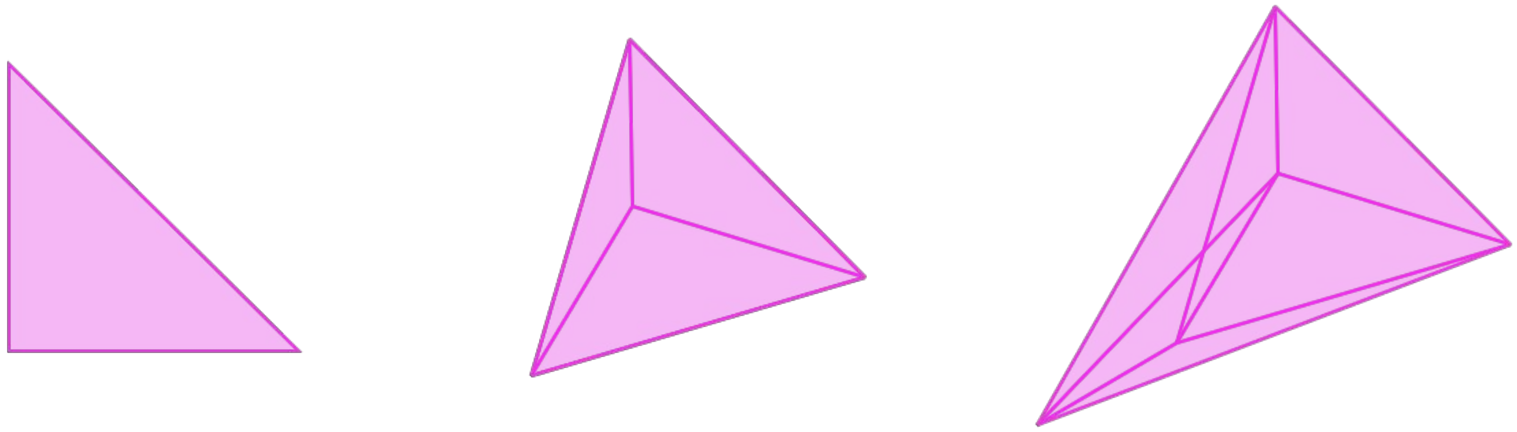
Standard Simplex (in 2D)

Take the convex hull of:

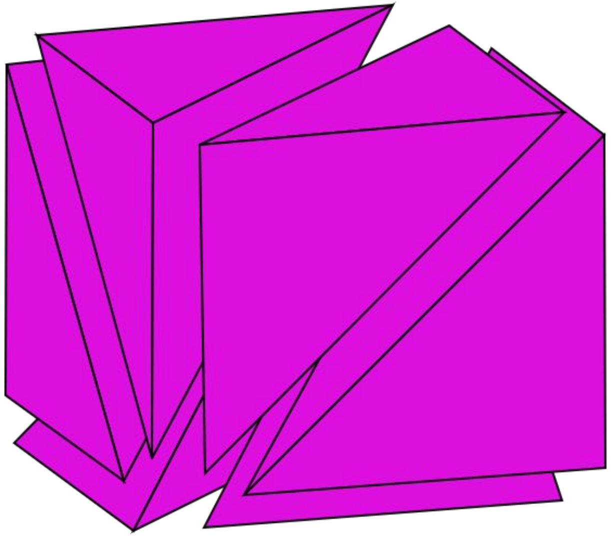
- (0,0)
- (1,0)
- (0,1)



Simplices…in any dimension!



Every polytope can be broken down into simplices.



Combinatorics is the art and science of distilling a complex mathematical structure into simple attributes and developing from this a deeper understanding of the original structure. - Josephine Yu (Georgia Tech)

What can we compute?

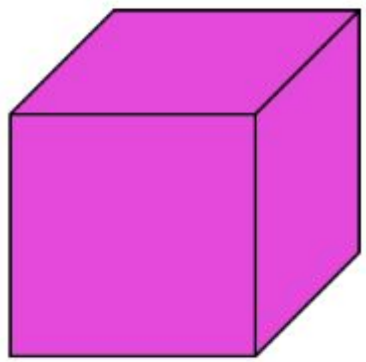
Why compute volume?

- Is an important piece of data in optimization problems.
- Computing volume can be hard, even for polytopes
- Shows up in other fields of math.

Discrete vs. continuous volume

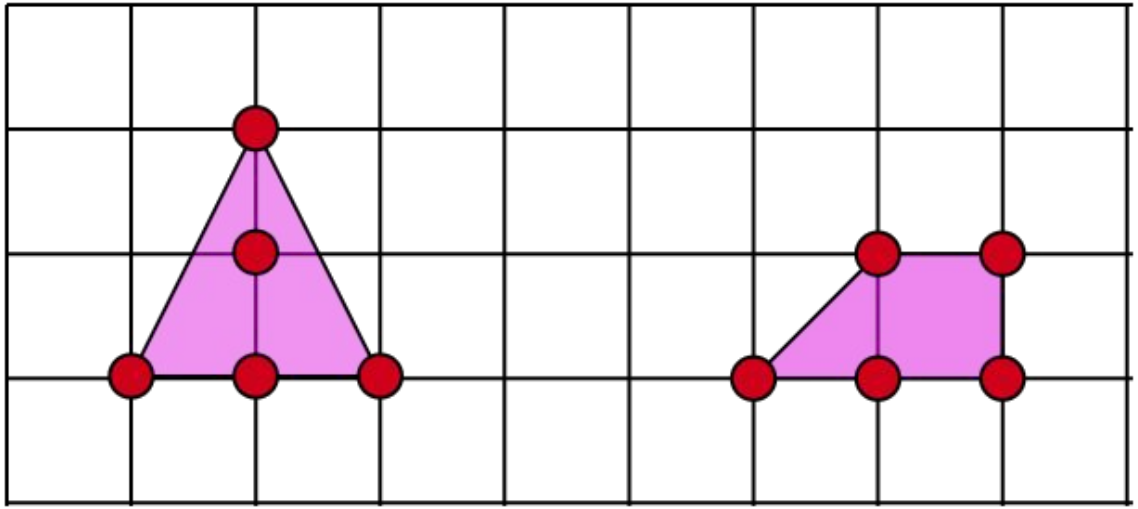
Continuous volume:

- What we usually think of volume
- Can compute using integrals



Discrete volume:

Two polygons with 5 lattice points

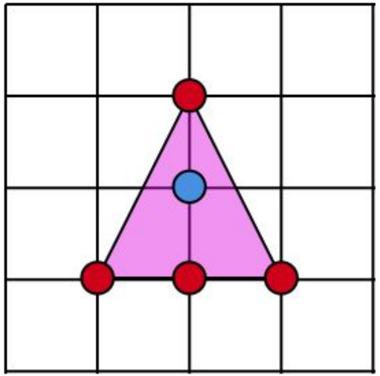


Case Study in 2D

Pick’s Theorem (Georg Alexander Pick, 1899):

- Consider a polygon whose vertices have integer coordinates.
- Let *B* be the number of integer points on the boundary, and *I* be the number of integer points on the interior.
- Then we can compute the area *A*:

$$A = I + \frac{1}{2}B - 1$$

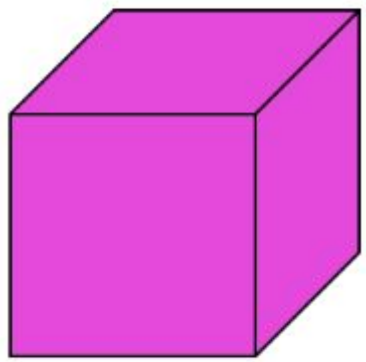


$$2 = 1 + \frac{1}{2}(4) - 1$$

Discrete vs. continuous volume

Continuous volume:

- What we usually think of volume
- Can compute using integrals



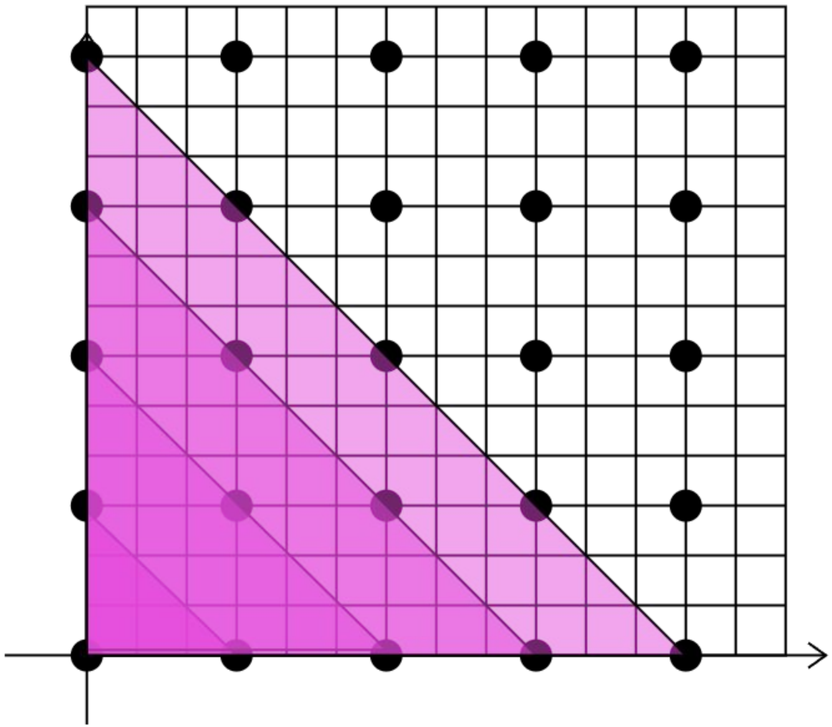
Discrete volume:

Discrete Volume

Lattice polytope: All vertices have integer coordinates

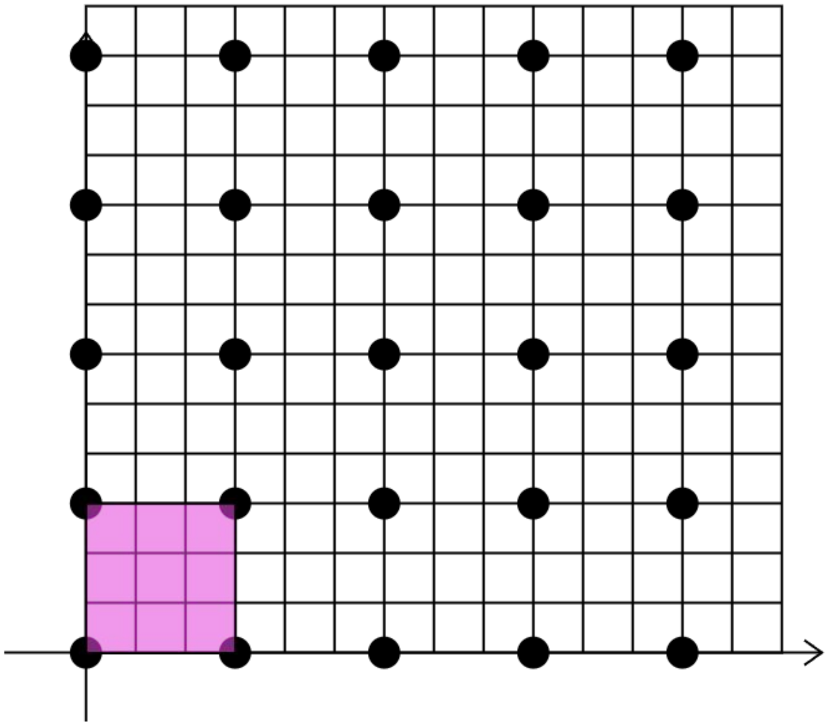
For a lattice polytope P, we define its discrete volume as:

$$\mathrm{ehr}_P(n) = |nP \cap \mathbb{Z}^d|$$



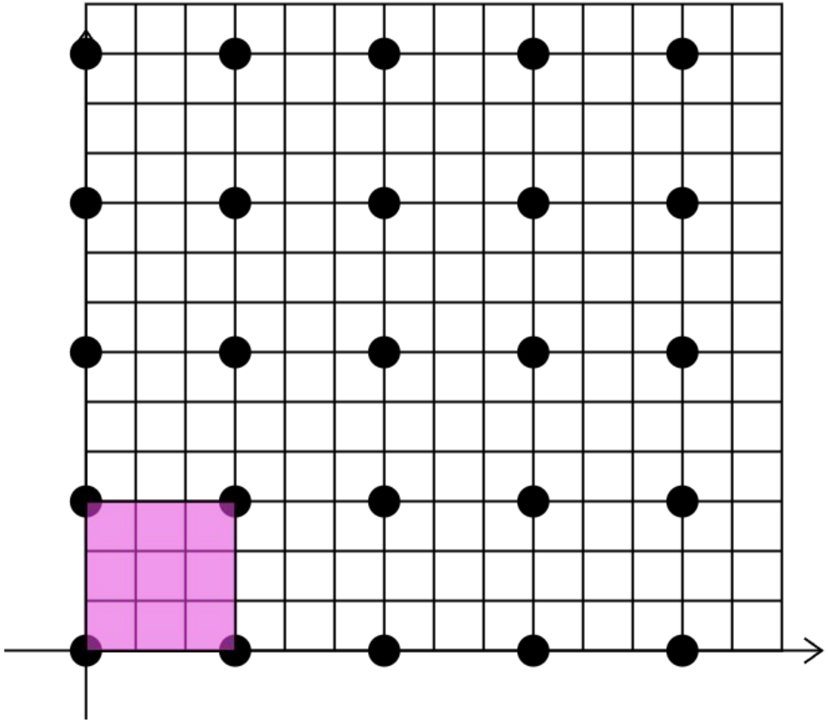
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	
2	
3	
4	



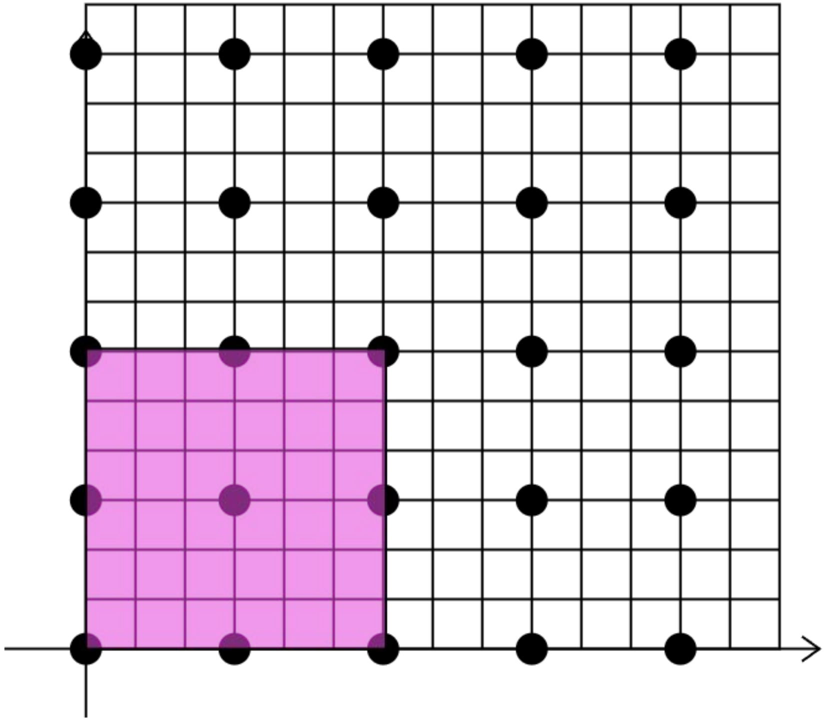
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	
3	
4	



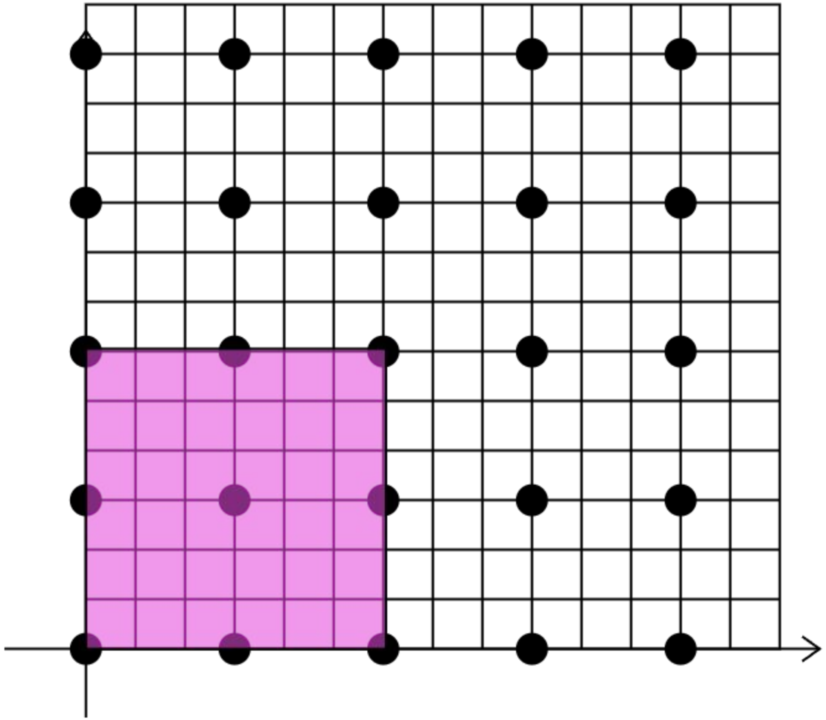
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	
3	
4	



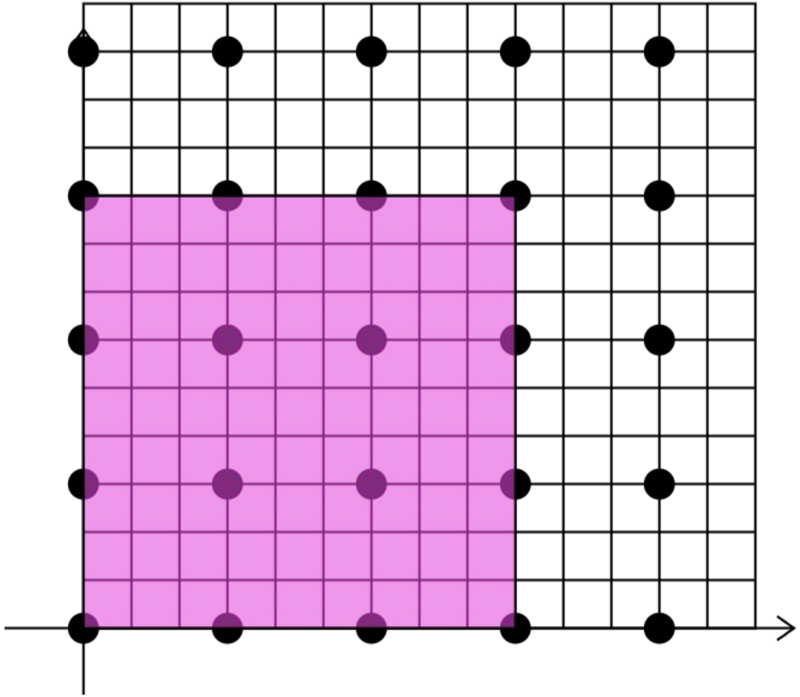
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	9
3	
4	



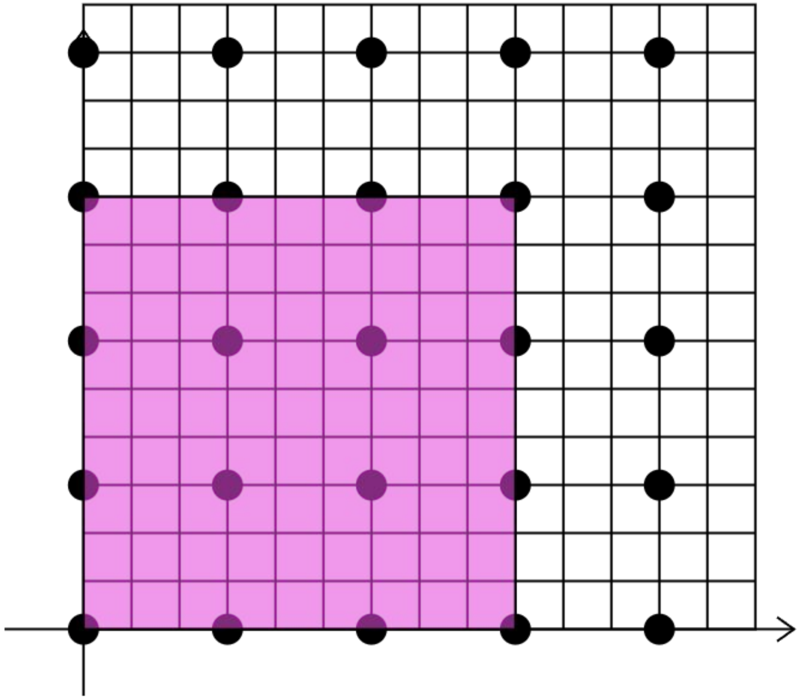
Discrete volume of an old friend…

n	$\text{ehr}_P(n)$
1	4
2	9
3	
4	



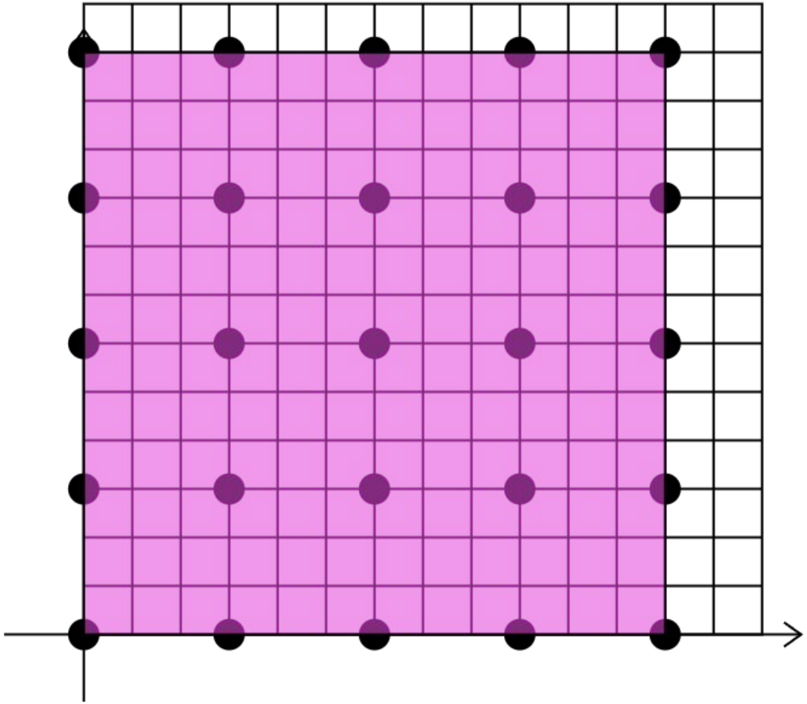
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	9
3	16
4	



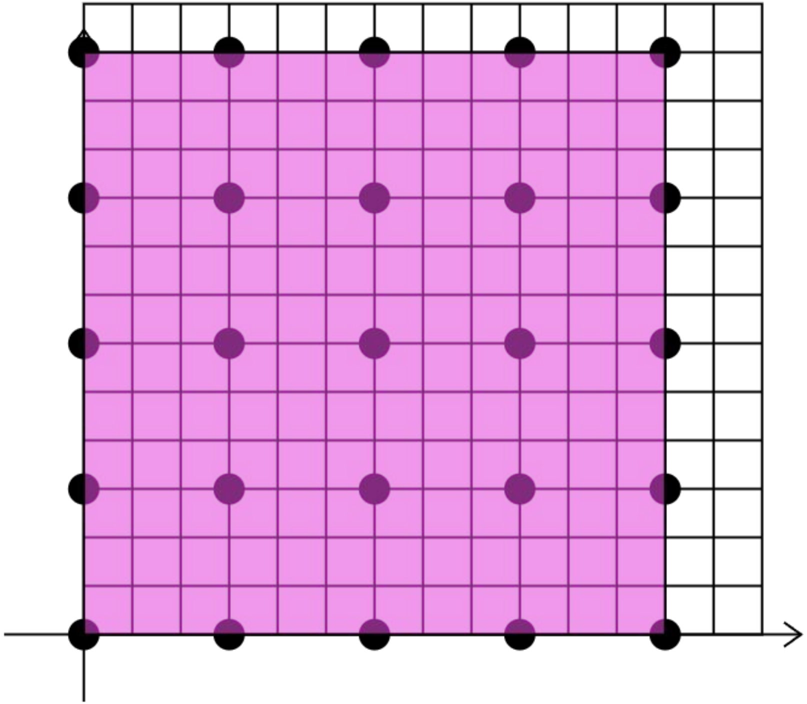
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	9
3	16
4	



Discrete volume of an old friend…

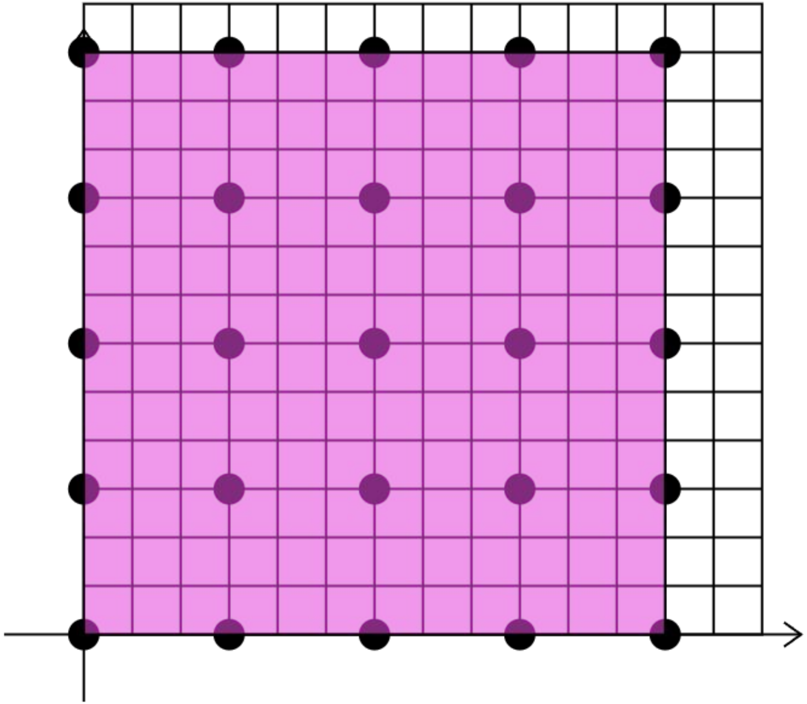
n	$\mathrm{ehr}_P(n)$
1	4
2	9
3	16
4	25



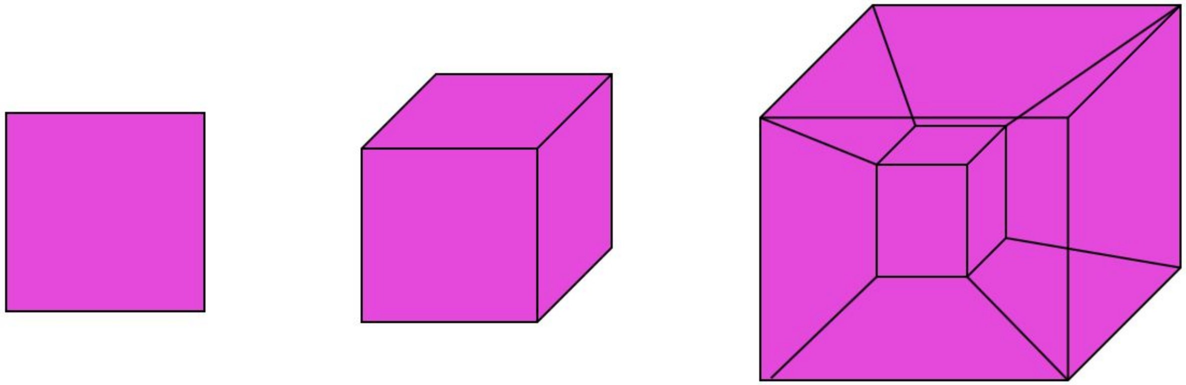
Discrete volume of an old friend…

n	$\mathrm{ehr}_P(n)$
1	4
2	9
3	16
4	25

$\mathrm{ehr}_P(n) = (n+1)^2 = n^2+2n+1$



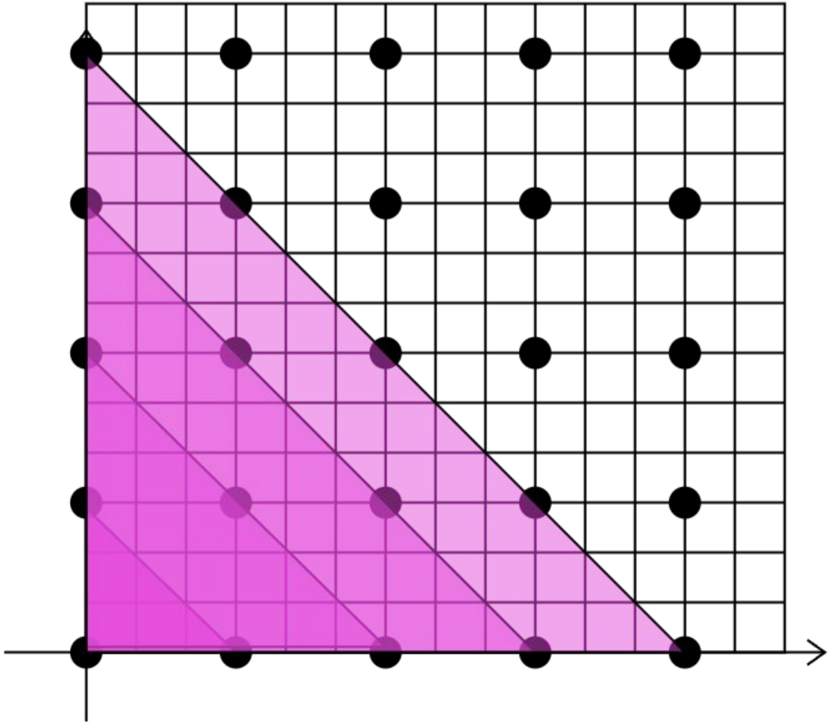
Discrete volume of d-dimensional unit cube



$\mathrm{ehr}_P(n) = (n+1)^d$

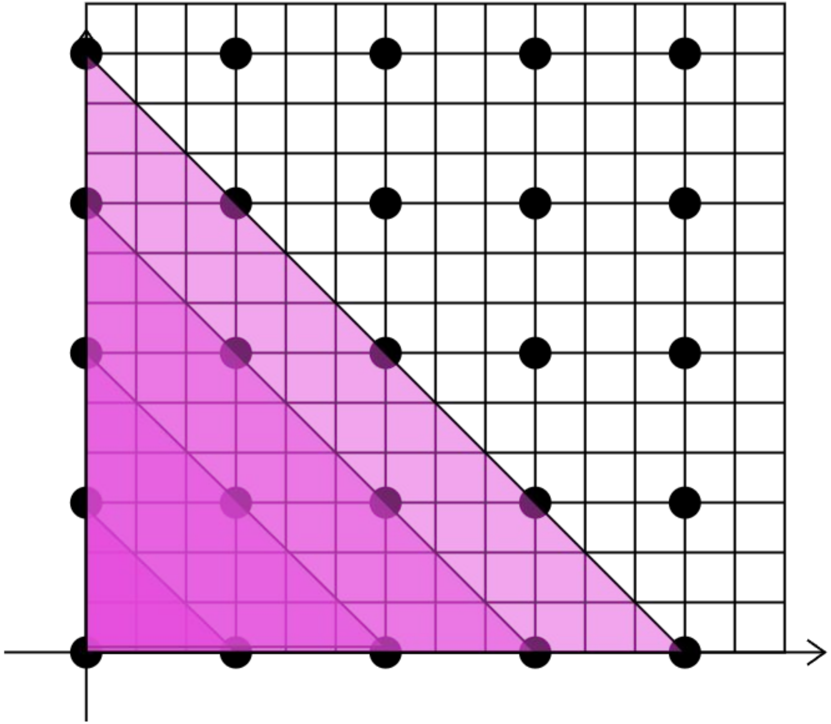
Discrete volume of another old friend…

n	$\mathrm{ehr}_P(n)$
1	
2	
3	
4	



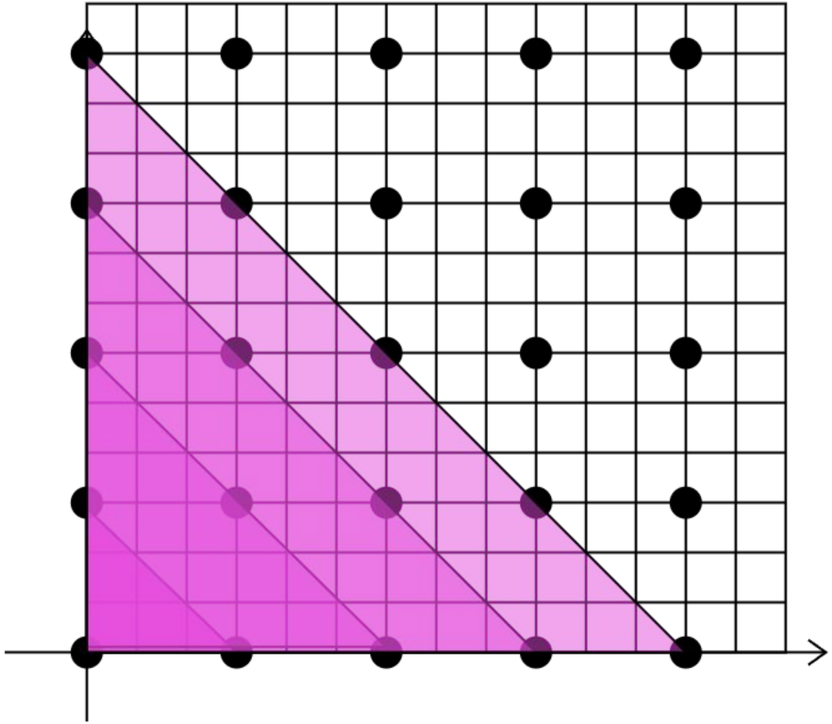
Discrete volume of another old friend…

n	$\mathrm{ehr}_P(n)$
1	3
2	
3	
4	



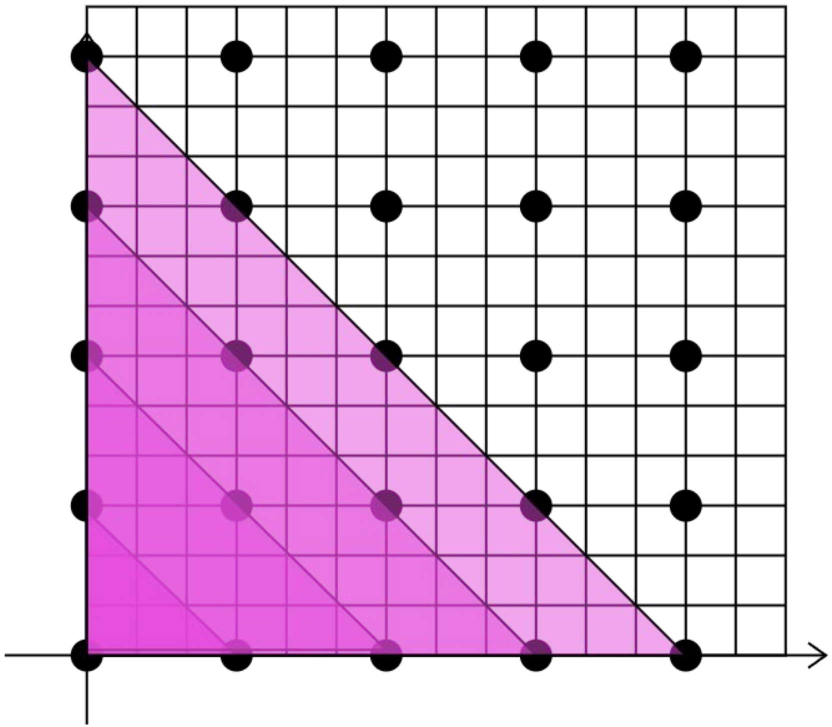
Discrete volume of another old friend…

n	$\text{ehr}_P(n)$
1	3
2	6
3	
4	



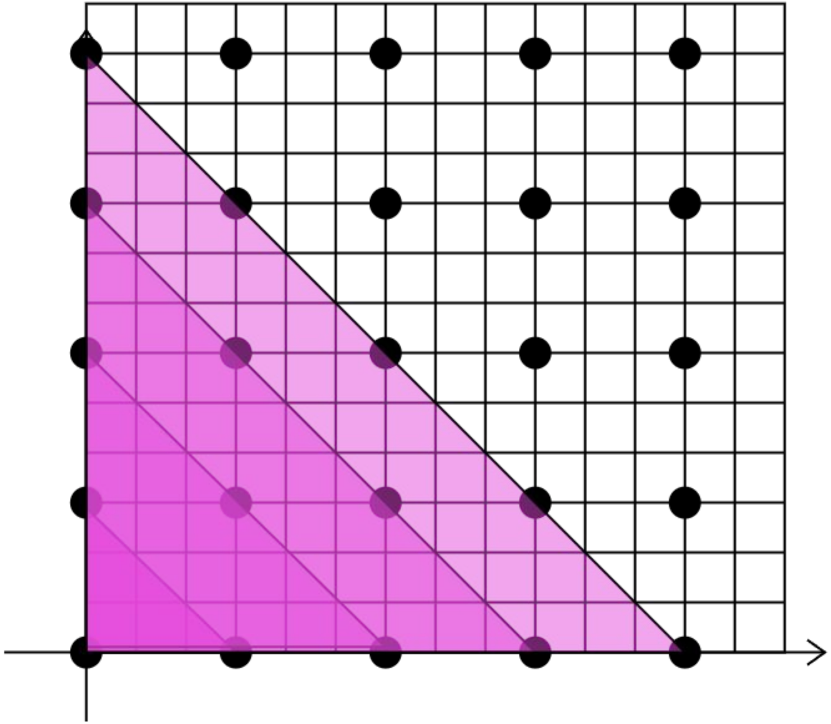
Discrete volume of another old friend…

n	$\mathrm{ehr}_P(n)$
1	3
2	6
3	10
4	



Discrete volume of another old friend…

n	$\mathrm{ehr}_P(n)$
1	3
2	6
3	10
4	15



What's that sequence?

The OEIS is supported by the many generous donors to the OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

3,6,10,15 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:3,6,10,15**

Displaying 1-10 of 268 results found. page 1 2 3 4 5 6 7 8 9 10 ... 27

Sort: relevance | references | number | modified | created Format: long | short | data

A000217 Triangular numbers: a(n) = binomial(n+1,2) = n*(n+1)/2 = 0 + 1 + 2 + ... + n. +30 4362

(Formerly M2535 N1002)

0, 1, **3, 6, 10, 15**, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431 (list; graph; refs; listen; history; text; internal format)

Main fact about discrete volume

- For a d-dimensional lattice polytope, the function $\mathrm{ehr}_P(n)$ is always a polynomial function.
- Proved in the 1960’s by Eugene Ehrhart – so we call it the Ehrhart polynomial.
- Why is this cool?

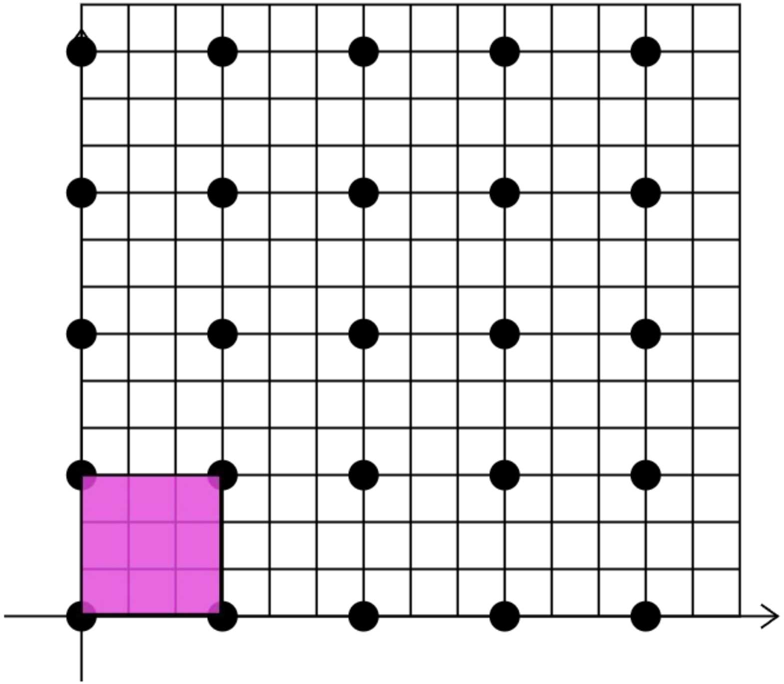
Anatomy of Ehrhart polynomial

$$ehr_P(n) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0$$

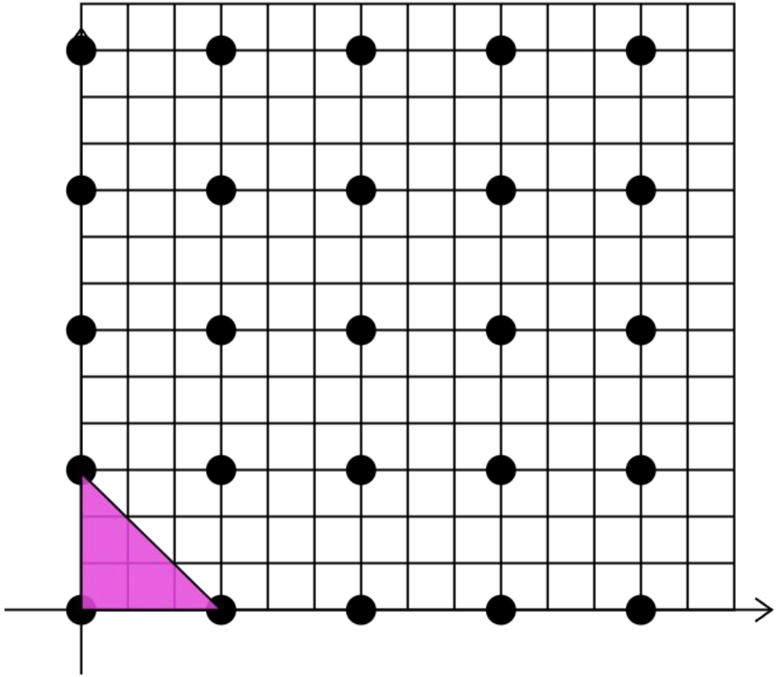
continuous volume!

always = 1

Let's try it out!



$$ehr_P(n) = n^2 + 2n + 1$$



$$ehr_P(n) = \frac{1}{2}n^2 + \frac{3}{2}n + 1$$

What research questions do we ask?

Categorization of Ehrhart Polynomials

$$ehr_P(n) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0$$

continuous volume!

always = 1

Computations of interesting Ehrhart polynomials

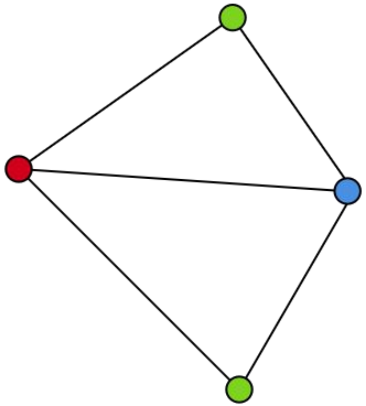
Some Student Work

Andrés R. Vindas Meléndez, John Lentfer, and Mitsuki Hanada

Bianca Teves, Josephine Brooks, Sophie Rubenfeld (and me!)

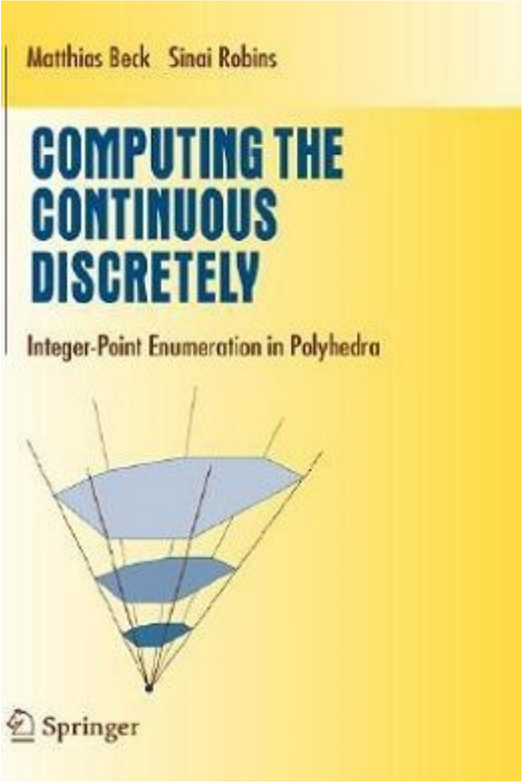
What other problems can we interpret using Ehrhart Theory?

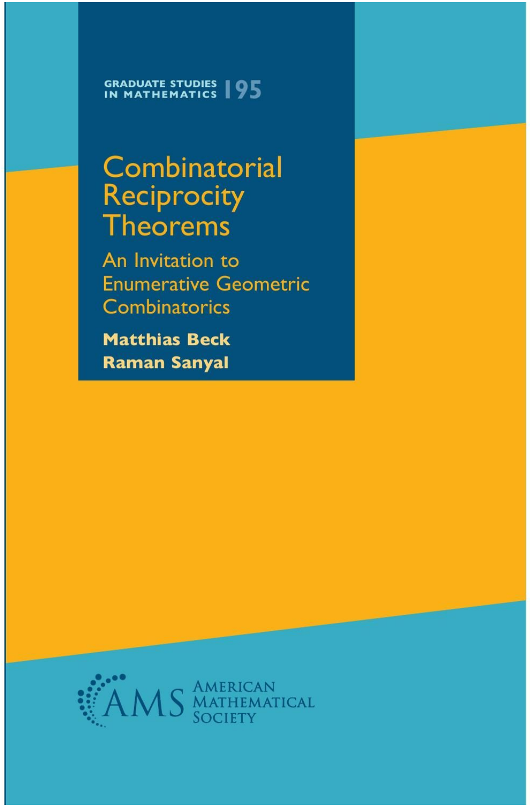
- An example in combinatorics: Coloring vertices of graphs



How many ways are there to color the vertices of a graph using *n* colors such that no two adjacent vertices share a color?

Some Friendly Resources





Thank you all!